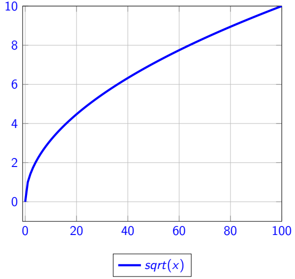
Chapter 3.11: Linearization and Differentials

Motivational Question

Example: Estimate $\sqrt{65}$ without using calculator.

$\boxed{\text{Approximate } \sqrt{x} \text{ using linear function!}}$



$f(x) = \sqrt{x}$
$f'(x) = \frac{1}{2}x^{-1/2}$
Tangent line at $x = 64$ that looks like

$$y = mx + b.$$

$m = f'(64) = \frac{1}{16}$
Evaluate b by $x = 64, y = 8$.
$8 = \frac{1}{16} \cdot 64 + b$
$b = 4$
So we are getting
$y = \frac{1}{16} \cdot 65 + 4 = 8 + \frac{1}{16}$.
Approximation of $\sqrt{65}$ is 8.0625000000
$\sqrt{65} = 8.0622577482$

Linearization

If $y = f(x)$ is differentiable at $x = a$, then

$$\boxed{L(x) = f(a) + f'(a)(x - a)}$$

is the line tangent to f at the point $(a, f(a))$. We call this the *linearization* of $f(x)$ at $x = a$.

Example: Find the linearization of the following functions at the specified point:

- $f(x) = x^3 - 2x + 3$ at $a = 2$
  $f'(x) = 3x^2 - 2$ hence $f'(2) = 10$. Also $f(2) = 7$. Therefore,
  $$L(x) = 7 + 10(x - 2)$$
- $f(x) = x + 1/x$ at $a = 1$
  $f'(x) = 1 - 1/x^2$ hence $f'(1) = 0$ Also $f(1) = 2$. Therefore
  $$L(x) = 2$$
- $f(x) = \tan(x)$ at $a = \pi$ As $f'(x) = \sec^2(x)$, $f(\pi) = 0$ and $f'(\pi) = 1$, the linearization is
  $$L(x) = x - \pi$$

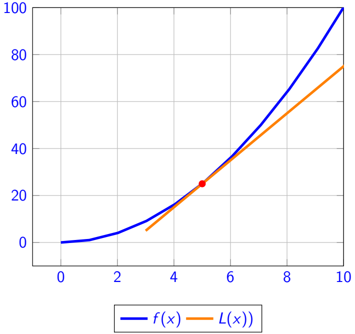
Approximation Examples $\boxed{L(x) = f(a) + f'(a)(x-a)}$

Find approximations for the following values:

1. $\sqrt{1.1}$ (The exact value is ≈ 1.0488088481).
   Let $f(x) = \sqrt{x}$ and $a = 1$. Since $f'(x) = \frac{1}{2\sqrt{x}}$, we have $f'(1) = 1/2$. As $f(1) = 1$, the linearization at $a = 1$ is
   $$L(x) = 1 + \frac{1}{2}(x-1)$$
   Consequently,
   $$\sqrt{1.1} \approx L(1.1) = 1 + \frac{1}{2}(.1) = 1 + \frac{1}{20} = \frac{21}{20} = 1.05$$

2. $\sin(.01)$ (The exact value is ≈ 0.00999983.)
   Let $f(x) = \sin(x)$ and $a = 0$. As $f'(x) = \cos(x)$, it must be that $f'(0) = 1$. Since $f(0) = 0$, the linearization at $a = 0$ is
   $$L(x) = x$$
   Consequently,
   $$\sin(.01) \approx L(.01) = .01$$

Approximation and Small Changes



Recall
$L(x) = f(a) + f'(a)(x - a)$.

Let $L(x)$ be a linearization of $f(x)$ at point a.

If $x = a$ is changed by Δx, what is change ΔL in $L(x)$?

$$\begin{aligned}\Delta L &= L(a + \Delta x) - L(a) \\ &= [f(a) + f'(a)\Delta x] - f(a) \\ &= f'(a)\Delta x\end{aligned}$$

Now treat Δx as dx and ΔL as dy. We obtain

$$dy = f'(a)dx$$

And by dividing by dx we get

$$\frac{dy}{dx} = f'(a).$$

Looks like a derivative...

$$f(a + dx) \approx f(a) + dy$$

Differentials

Leibniz says that we can treat $\frac{dy}{dx}$ like a fraction and write a formula for dy in terms of dx and $f'(x)$. This is known as the *differential* form of the derivative.

Find the differential form of the derivative of the following (use dy and dx for derivative of x and y):

1. $y = \sqrt{1 - x^2}$

$$dy = \frac{-x}{\sqrt{1 - x^2}}\, dx$$

2. $xy^2 - 4x^{3/2} - y = 0$

$$\begin{aligned} y^2\, dx + 2xy\, dy - 6\sqrt{x}\, dx - dy &= 0 \\ (2xy - 1)\, dy &= (6\sqrt{x} - y^2)\, dx \\ dy &= \frac{6\sqrt{x} - y^2}{2xy - 1}\, dx \end{aligned}$$

3. Use implicit differentiation on $xy^2 - 4x^{3/2} - y = 0$

$$y^2 + 2xy\frac{dy}{dx} - 6\sqrt{x} - \frac{dy}{dx} = 0 \qquad (2xy - 1)\frac{dy}{dx} = (6\sqrt{x} - y^2)$$

Approximation With Differentials Example

1. Approximate $\sqrt[3]{1.009}$ using differentials and $f(a+dx) \approx f(a) + dy$
   Let $f(x) = \sqrt[3]{x}$. Use $a = 1$ and $dx = 0.009$ Recall $dy = f'(a)dx$. Note $f'(x) = 3(x)^{2/3}$.

$$\begin{aligned} f(1.009) &= f(a+dx) \approx f(a) + dy \\ &= f(a) + f'(a)dx = \sqrt[3]{1} + 31^{2/3}(0.009) = 1 + 0.003 \\ &= 1.003 \end{aligned}$$

2. Approximate $\sqrt[3]{1.009}$ using $L(x) = f(a) + f'(a)(x-a)$.
   Let $f(x) = \sqrt[3]{x}$ and $a = 1$. Since $f'(x) = (1/3)x^{-2/3}$, we have that $f'(1) = 1/3$. As $f(1) = 1$, the linearization at $a = 1$ is

$$L(x) = 1 + (1/3)(x-1)$$

   Plugging in $x = 1.009$ yields

$$\sqrt[3]{1.009} \approx L(1.009) = 1 + (1/3)(.009) = 1 + \frac{1}{3} \cdot \frac{9}{1000} = 1 + \frac{3}{1000} = 1.003$$

$\sqrt[3]{1.009} = 1.0029910447317696162055702742437385366236726998841849434\ldots$